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Comparing Two Integral Means for Absolutely Continuous Mappings Whose Derivatives are in $L_\infty[a, b]$ and Applications

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Abstract—Estimates of the difference of two integral means on $[a, b]$, $[c, d]$ with $[c, d] \subset [a, b]$ in terms of the sup norm of the derivative and applications for pdfs, special means, Jeffreys' divergence, and continuous streams are given.

Keywords—Ostrowski's inequality, Integral means, Special means, Probability density function, Jeffreys divergence, Continuous streams.

1. INTRODUCTION

In 1938, Ostrowski proved the following integral inequality [1].

THEOREM 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and assume that $|f'(x)| \le M$ for all $x \in (a, b)$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} \left(\frac{x - (a+b)/2}{b-a} \right)^2 \right] (b-a)M, \tag{1.1}$$

for all $x \in [a, b]$. The constant $1/4$ is the best possible.

For some generalisations and related results, see the book [2, pp. 468–484], the papers [3–16], and the website http://rgmia.vu.edu.au/ where many papers devoted to this inequality can be accessed on-line.

We note that if we use the easily verified identity [2, p. 585], which also holds for absolutely continuous mappings $f : [a, b] \to \mathbb{R}$,

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t)\, dt, \qquad x \in [a, b], \tag{1.2}$$

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where the kernel $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a, & \text{if } a \le t \le x \le b, \\ t-b, & \text{if } a \le x < t \le b, \end{cases}$$

and if we assume that $f' \in L_\infty[a,b]$ and $\|f'\|_\infty := \operatorname{ess\,sup}_{t\in[a,b]} |f'(t)|$, then we can replace M from (1.1) with $\|f'\|_\infty$.

For generalisations of (1.1), see [17,18], as well as the recent papers produced by the RGMIA members.

In this paper, we compare the two integral means

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du, \qquad [c,d] \subset [a,b],$$

where f is assumed to be absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$. Applications for probability density functions (pdfs) in probability theory, for special means including identric and logarithmic means, for Jeffreys' divergence in information theory, and for the sampling of continuous streams are also given.

2. SOME ANALYTIC INEQUALITIES

We start with the following identity which is of interest in itself.

LEMMA 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping and $a \le c < d \le b$. Denote $K_{c,d} : [a,b] \to \mathbb{R}$ the kernel given by*

$$K_{c,d}(s) := \begin{cases} \dfrac{a-s}{b-a}, & \text{if } s \in [a,c], \\ \dfrac{s-c}{d-c} + \dfrac{a-s}{b-a}, & \text{if } s \in (c,d), \\ \dfrac{b-s}{b-a}, & \text{if } s \in [d,b]. \end{cases} \tag{2.1}$$

Then we have the representation

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du = \int_a^b K_{c,d}(s) f'(s)\,ds. \tag{2.2}$$

PROOF. Using the integration by parts formula, we have

$$\begin{aligned}
\int_a^b K_{c,d}(s) f'(s)\,ds &= \int_a^c \left(\frac{a-s}{b-a}\right) f'(s)\,ds \\
&\quad + \int_c^d \left(\frac{s-c}{d-c} + \frac{a-s}{b-a}\right) f'(s)\,ds + \int_d^b \left(\frac{b-s}{b-a}\right) f'(s)\,ds \\
&= \frac{a-c}{b-a} f(c) + \frac{1}{b-a}\int_a^c f(s)\,ds + \left(1 + \frac{a-d}{b-a}\right) f(d) - \frac{a-c}{b-a} f(c) \\
&\quad - \left(\frac{1}{d-c} - \frac{1}{b-a}\right)\int_c^d f(s)\,ds - \frac{b-d}{b-a} f(d) + \frac{1}{b-a}\int_d^b f(s)\,ds \\
&= \frac{1}{b-a}\int_a^c f(s)\,ds + \frac{1}{b-a}\int_c^d f(s)\,ds + \frac{1}{b-a}\int_d^b f(s)\,ds - \frac{1}{d-c}\int_c^d f(s)\,ds \\
&= \frac{1}{b-a}\int_a^b f(s)\,ds - \frac{1}{d-c}\int_c^d f(s)\,ds,
\end{aligned}$$

and the identity (2.2) is proved. ∎

The following estimation result holds.

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := \operatorname*{ess\ sup}_{t\in[a,b]} |f'(t)| < \infty.$$

Then for $a \le c < d \le b$, we have the inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du\right| \\
&\le \left\{\frac{1}{4} + \left[\frac{(a+b)/2-(c+d)/2}{(b-a)-(d-c)}\right]^2\right\}[(b-a)-(d-c)]\,\|f'\|_\infty \\
&\le \frac{1}{2}[(b-a)-(d-c)]\,\|f'\|_\infty .
\end{aligned} \tag{2.3}$$

The constant 1/4 is best possible in the first inequality and 1/2 is best in the second one.

PROOF. Taking the modulus in (2.2), we may write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du\right| \\
&\le \int_a^b |K_{c,d}(s)|\,|f'(s)|\,ds \le \|f'\|_\infty \int_a^b |K_{c,d}(s)|\,ds \\
&= \|f'\|_\infty\left[\frac{1}{b-a}\int_a^c (s-a)\,ds + \int_c^d \left|\frac{s-c}{d-c}+\frac{a-s}{b-a}\right| ds + \frac{1}{b-a}\int_c^d (b-s)\,ds\right] := K.
\end{aligned} \tag{2.4}$$

However,

$$\int_a^c (s-a)\,ds = \frac{(c-a)^2}{2}, \qquad \int_d^b (b-s)\,ds = \frac{(b-d)^2}{2},$$

and

$$\begin{aligned}
L &:= \int_c^d \left|\frac{s-c}{d-c}+\frac{a-s}{b-a}\right| ds \\
&= \frac{1}{(b-a)(d-c)}\int_c^d |[(b-a)-(d-c)]\,s - cb + ad|\,ds.
\end{aligned}$$

Consider the affine mapping

$$g(s) := [(b-a)-(d-c)]\,s - cb + ad.$$

As $b-a > d-c$, we get $g(s_0) = 0$ iff $s_0 = (cb-ad)/((b-a)-(d-c))$. Simple calculation proves that $s_0 \in [c,d]$, and then

$$\begin{aligned}
&\int_c^d |[(b-a)-(d-c)]\,s - cd + ab|\,ds \\
&= [(b-a)-(d-c)]\int_c^d |s-s_0|\,ds \\
&= [(b-a)-(d-c)]\left[\int_c^{s_0} (s_0-s)\,ds + \int_{s_0}^d (s-s_0)\,ds\right] \\
&= [(b-a)-(d-c)]\left[\frac{(s_0-c)^2}{2} + \frac{(d-s_0)^2}{2}\right].
\end{aligned}$$

However,

$$s_0 - c = \frac{(c-a)(d-c)}{(b-a)-(d-c)}$$

and

$$d - s_0 = \frac{(d-c)(b-d)}{(b-a)-(d-c)},$$

and so

$$\begin{aligned} L &= \frac{1}{(b-a)(d-c)} \cdot \frac{[(b-a)-(d-c)]}{2} \left[\frac{(c-a)^2(d-c)^2}{[(b-a)-(d-c)]^2} + \frac{(d-c)^2(b-d)^2}{[(b-a)-(d-c)]^2} \right] \\ &= \frac{(d-c)}{(b-a)\,[(b-a)-(d-c)]} \left[\frac{(c-a)^2}{2} + \frac{(d-c)^2}{2} \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} K &= \|f'\|_\infty \left[\frac{(c-a)^2}{2(b-a)} + \frac{(b-d)^2}{2(b-a)} + \frac{(d-c)}{(b-a)} \cdot \frac{1}{(b-a)(d-c)} \left[\frac{(c-a)^2}{2} + \frac{(b-d)^2}{2} \right] \right] \\ &= \|f'\|_\infty \frac{(c-a)^2 + (b-d)^2}{2(b-a)} \left[1 + \frac{d-c}{[(b-a)-(d-c)]^2} \right] \\ &= \frac{\|f'\|_\infty}{(b-a)} \left[\frac{[(b-a)-(d-c)]^2}{4} + \left(\frac{a+b}{2} - \frac{c+d}{2} \right)^2 \right] \frac{(b-a)}{[(b-a)-(d-c)]} \\ &= \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{b-a-(d-c)} \right)^2 \right] [(b-a)-(d-c)]\,\|f'\|_\infty, \end{aligned}$$

and the first part of inequality (2.3) is proved.

To prove the last part of (2.3), we observe that, by a simple computation

$$\left(\frac{a+b}{2} - \frac{c+d}{2} \right)^2 \le \frac{1}{4}\,[(b-a)-(d-c)]^2$$

is equivalent with

$$(c-a)(b-d) \ge 0,$$

which is obvious by the selection of a, b, c, d.

Taking into account that $K_{c,d}$ is negative on $[a, s_0]$ and positive on $[s_0, b]$, then we can conclude that the functions $f_0(s) := A|s - s_0|$ $(A > 0)$ are the extremals in (2.3), and the constant $1/4$ is the best possible in the first inequality in (2.3). The fact that $1/2$ is the best constant in the second inequality is obvious. ∎

REMARK 1. The above inequality (2.3) may be regarded as a generalisation of the classical Ostrowski inequality.

Indeed, if we assume that $c = x \in (a,b)$ and $d = x + \varepsilon$, ε is such that $x + \varepsilon \in (a,b)$, then by (2.3), we get

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(u)\,du \right| \le \left[\frac{1}{4} + \frac{((a+b)/2 - x - \varepsilon/2)^2}{(b-a-\varepsilon)^2} \right] [(b-a)-\varepsilon]\,\|f'\|_\infty. \quad (2.5)$$

Now, letting $\varepsilon \to 0+$ and taking into account, by the continuity of f at x, that we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(u)\,du = f(x),$$

then by (2.5), we may deduce that

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \le \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty, \quad (2.6)$$

which is Ostrowski's inequality.

COROLLARY 1. *Assume that a, b, c, d are as in Theorem 2. Then, the best inequality we can get from (2.3) is the one for $(a+b)/2 = (c+d)/2$, i.e.,*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(u)\,du \right| \leq \frac{1}{4} \left[(b-a) - (d-c) \right] \|f'\|_{\infty} . \tag{2.7}$$

The constant 1/4 is the best possible.

Now, for any $x \in (a,b)$, we can find a $\delta > 0$ such that the mapping $F(x,\cdot) : [-\delta,\delta] \to \mathbb{R}$, defined by

$$F(x,t) := \frac{1}{t} \int_{x-t/2}^{x+t/2} f(u)\,du, \tag{2.8}$$

is well defined.

We can prove the following corollary.

COROLLARY 2. *Assume that the mapping $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $f' \in L_{\infty}[a,b]$. Then, for any $x \in (a,b)$, the mapping $F(x,\cdot)$ is locally Lipschitzian and the Lipschitzian constant is $(1/4)\|f'\|_{\infty}$ and is independent of x.*

PROOF. As $x \in (a,b)$, then there exists a $\delta > 0$ such that $x+t/2$, $x-t/2 \in (a,b)$ for all $t \in [-\delta,\delta]$, and the mapping (2.8) is well defined.

Assume that $t_1, t_2 \in [-\delta,\delta]$ and $t_2 > t_1$. Then $[x+t_2/2,\ x-t_2/2] \supset [x+t_1/2,\ x-t_1/2]$, and if we apply Theorem 2 on these intervals, we obtain

$$\left| \frac{1}{t_2} \int_{x-t_2/2}^{x+t_2/2} f(u)\,du - \frac{1}{t_1} \int_{x-t_1/2}^{x+t_1/2} f(u)\,du \right| \leq \frac{1}{4} (t_2 - t_1) \|f'\|_{\infty} ,$$

which shows that

$$|F(x,t_2) - F(x,t_1)| \leq \frac{1}{4} (t_2 - t_1) \|f'\|_{\infty} .$$

If $t_2 < t_1$, a similar inequality applies, and then, we may conclude that for any $t_1, t_2 \in [-\delta,\delta]$, we have

$$|F(x,t_2) - F(x,t_1)| \leq \frac{1}{4} \|f'\|_{\infty} |t_2 - t_1| ,$$

which proves the corollary. ∎

3. APPLICATIONS FOR PDFS

Assume that $f : [a,b] \to \mathbb{R}_+$ is a probability density function (pdf) of a certain random variable X and $F : [a,b] \to \mathbb{R}_+$, $F(x) = \int_a^x f(x)\,dx$ is its cumulative distribution function. Then we can state the following proposition.

PROPOSITION 1. *With the previous assumptions for f and F, we have*

$$\left| F(x) - \frac{x-a}{b-a} \right| \leq \frac{1}{2} (b-x)(x-a) \|f'\|_{\infty} , \qquad x \in [a,b], \tag{3.1}$$

provided that $f' \in L_{\infty}[a,b]$.

PROOF. If we choose $c = a$ and $d = x$ in (2.3), we obtain the desired inequality. ∎

Another inequality for the mapping $F(\cdot)$ is embodied in the following proposition.

PROPOSITION 2. *Let f and F be as above and $f \in L_\infty[a,b]$. Then we have the inequality*

$$\left| \frac{(b - E(X))(x-a)}{b-a} + E_x(X) - xF(x) \right| \le \frac{1}{2}(b-x)(x-a)\|f\|_\infty, \tag{3.2}$$

for all $x \in [a,b]$, where

$$E_x(X) := \int_a^x u f(u)\,du, \qquad x \in [a,b].$$

PROOF. If we write inequality (2.3) for F, we get

$$\left| \frac{1}{b-a}\int_a^b F(t)\,dt - \frac{1}{x-a}\int_a^x F(u)\,du \right| \le \frac{1}{2}(b-x)\|f\|_\infty. \tag{3.3}$$

However,

$$\int_a^b F(t)\,dt = b - E(X)$$

and

$$\int_a^x F(u)\,du = xF(x) - \int_a^x u f(u)\,du = xF(x) - E_x(X).$$

Now, by (3.3) we deduce (3.2). ∎

Let us consider the *Beta function*

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad p,q > -1,$$

and the *incomplete Beta function*

$$B(x;p,q) := \int_0^x t^{p-1}(1-t)^{q-1}\,dt.$$

If we define $f(t) := t^{p-1}(1-t)^{q-1}$, we observe that for either $p \in (0,1)$ or $q \in (0,1)$, we have

$$\|f\|_\infty := \sup_{t\in(0,1)} f(t) = +\infty.$$

Assume that $p, q \ge 1$. Then

$$\frac{df(t)}{dt} = t^{p-2}(1-t)^{q-1}\left[-(p+q-2)t + (p-1)\right].$$

We observe that

$$\frac{df(t)}{dt} = 0$$

iff $t_0 = (p-1)/p+q-2) \in (0,1)$ $(p,q>1)$ and then $\frac{df(t)}{dt} > 0$ on $(0,t_0)$ and $\frac{df(t)}{dt} < 0$ on $(t_0,1)$. Consequently,

$$\|f\|_\infty = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}.$$

Consider now the random variable X having the pdf $\rho(t) := f(t)/B(p,q)$, $t \in (0,1)$; then,

$$E(X) = \frac{1}{B(p,q)}\int_0^1 t^p(1-t)^{q-1}\,dt = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}.$$

Also, we have

$$E_x(X) = \int_0^x \frac{tf(t)}{B(p,q)}\,dt = \frac{\int_0^x t^p(1-t)^{q-1}\,dt}{B(p,q)} = \frac{B(x;p+1,q)}{B(p,q)}.$$

Using Proposition 2, we may state the following proposition.

PROPOSITION 3. *Let X be a Beta random variable with the parameters (p,q), $p,q \ge 1$. Then we have the inequality*

$$\left| B(x;p+1,q) - xB(x;p,q) + \frac{qx}{p+q}B(p,q) \right| \le \frac{1}{2}(1-x)x \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)} \cdot B(p,q),$$

for all $x \in [0,1]$.

4. APPLICATION FOR SPECIAL MEANS

Let us recall the following means for two positive numbers.

(1) *The logarithmic mean*

$$L = L(a,b) := \begin{cases} a, & \text{if } a = b, \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b, \end{cases} \qquad a, b > 0.$$

(2) *The identric mean*

$$I = I(a,b) := \begin{cases} a, & \text{if } a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & \text{if } a \neq b, \end{cases} \qquad a, b > 0.$$

(3) *The p-logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & \text{if } a \neq b, \end{cases} \qquad a, b > 0.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$. The following examples illustrate the bounds developed in Section 2 involving the difference of integral means over different intervals.

(1) Consider the mapping $f : (0, \infty) \to (0, \infty)$, $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$. Then for $0 < a \leq c < d \leq b < \infty$, we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt = L_p^p(a,b), \qquad \frac{1}{d-c}\int_c^d f(t)\,dt = L_p^p(c,d),$$

and

$$\|f'\|_{\infty,[a,b]} = \delta_p(a,b) = \begin{cases} pb^{p-1}, & \text{if } p \geq 1, \\ |p|a^{p-1}, & \text{if } p \in (-\infty, 1) \setminus \{-1, 0\}. \end{cases}$$

Using inequality (2.3), we have

$$\left|L_p^p(a,b) - L_p^p(c,d)\right| \leq \frac{1}{4}\left[1 + \left(\frac{(b-c)-(d-a)}{b-a-(d-c)}\right)^2\right][(b-a)-(d-c)]\,\delta_p(a,b). \tag{4.1}$$

(2) Consider the mapping $f : (0, \infty) \to (0, \infty)$, $f(x) = 1/x$, and $0 < a \leq c < d \leq b < \infty$. We have

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{L(a,b)}, \qquad \frac{1}{d-c}\int_c^d f(t)\,dt = \frac{1}{L(c,d)},$$

and

$$\|f'\|_{\infty,[a,b]} = \frac{1}{a^2}.$$

Using inequality (2.3), we may state that

$$|L(a,b) - L(c,d)| \leq \frac{1}{4}\left[1 + \left(\frac{(b-c)-(d-a)}{b-a-(d-c)}\right)^2\right][(b-a)-(d-c)]\,\frac{L(a,b)L(c,d)}{a^2}. \tag{4.2}$$

(3) Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$, and $0 < a \leq c < d \leq b < \infty$. We have

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \ln I(a,b), \qquad \frac{1}{d-c}\int_c^d f(t)\,dt = \ln I(c,d),$$

and

$$\|f'\|_{\infty,[a,b]} = \frac{1}{a}.$$

Using inequality (2.3), we may write

$$\left|\ln\left[\frac{I(a,b)}{I(c,d)}\right]\right| \leq \frac{1}{4}\left[1 + \left(\frac{(b-c)-(d-a)}{b-a-(d-c)}\right)^2\right]\frac{[(b-a)-(d-c)]}{a}. \tag{4.3}$$

5. APPLICATIONS TO JEFFREYS' DIVERGENCE IN INFORMATION THEORY

Assume that a set χ and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\Omega := \{p \mid p : \chi \to \mathbb{R},\ p(x) \geq 0 \text{ and } \int_\chi p(x)\, d\mu(x) = 1\}$.

The Jeffreys distance D_J [19] is well known among the information divergences and is very useful in information theory. It is defined by

$$D_J(p,q) := \int_\chi [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \qquad p,q \in \Omega.$$

The following inequalities involving the Jeffreys divergence are known (see, for example, the book on-line by Taneja [20]):

$$D_{Ha}(p,q) \geq \exp\left[-\frac{1}{2} D_J(p,q)\right], \qquad p,q \in \Omega, \tag{5.1}$$

$$D_{Ha}(p,q) \geq 1 - \frac{1}{4} D_f(p,q), \qquad p,q \in \Omega, \qquad \text{and} \tag{5.2}$$

$$D_J(p,q) \geq 4\left[1 - D_B(p,q)\right], \qquad p,q \in \Omega, \tag{5.3}$$

where $D_{Ha}(\cdot,\cdot)$ is the *Harmonic distance*, namely

$$D_{Ha}(p,q) := \int_\chi \frac{2p(x)q(x)}{p(x)+q(x)}\, d\mu(x), \qquad p,q \in \Omega,$$

and $D_B(\cdot,\cdot)$ is the Bhattacharyya distance, which is given by

$$D_B(p,q) := \int_\chi \sqrt{p(x)q(x)}\, d\mu(x).$$

In the recent paper [21], the authors proved the following inequalities as well:

$$2D_\Delta(p,q) \leq D_J(p,q) \leq \frac{1}{2}\left[D_{\chi^2}(p,q) + D_{\chi^2}(q,p)\right], \qquad p,q \in \Omega, \tag{5.4}$$

$$0 \leq D_J(p,q) - 2D_\Delta(p,q) \leq \frac{1}{6} D_*(p,q), \tag{5.5}$$

and

$$0 \leq \frac{1}{2}\left[D_{\chi^2}(p,q) + D_{\chi^2}(q,p)\right] - D_J(p,q) \leq \frac{1}{6} D_*(p,q), \tag{5.6}$$

where $D_{\chi^2}(\cdot,\cdot)$ is the *chi-square divergence*, given by

$$D_{\chi^2}(p,q) := \int_\chi p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x),$$

$D_\Delta(\cdot,\cdot)$ is the *triangular discrimination* introduced by Topsoe in [22], namely,

$$D_\Delta(p,q) := \int_\chi \frac{[p(x) - q(x)]^2}{p(x)+q(x)}\, d\mu(x), \qquad p,q \in \Omega,$$

and $D_*(\cdot,\cdot)$ was introduced in [21],

$$D_*(p,q) := \int_\chi \frac{(p(x) - q(x))^4}{\sqrt{p^3(x)q^3(x)}}\, d\mu(x).$$

In this section, we are going to point out other inequalities for Jeffreys' divergence by the use of inequality (4.2) written in the following equivalent form:

$$\begin{aligned} \left|\frac{\ln b - \ln a}{b-a} - \frac{\ln d - \ln c}{d-c}\right| &\leq \frac{1}{4}\left[1 + \left(\frac{(b-c)-(d-a)}{b-a-|d-c|}\right)^2\right]\left[(b-a) - |d-c|\right] \cdot \frac{1}{a^2} \\ &= \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{b-a-|d-c|}\right)^2\right]\left[(b-a) - |d-c|\right] \cdot \frac{1}{a^2}, \end{aligned} \tag{5.7}$$

for all $c, d \in [a,b]$.

We may state the following proposition.

PROPOSITION 4. *Let $p, q \in \Omega$ with $0 < r \le q(x)/p(x) \le R < \infty$ for a.e. $x \in \chi$ $(r \le 1 \le R)$. Then we have the inequality*

$$\begin{aligned}\left| D_J(p,q) - \frac{1}{L(r,R)} D_{\chi^2}(p,q) \right| &\le \int_\chi \left[\frac{1}{4} + \left(\frac{((r+R)/2)p(x) - (p(x)+q(x))/2}{(R-r)p(x) - |q(x)-p(x)|} \right)^2 \right] \\ &\quad \times [(R-r)p(x) - |q(x)-p(x)|] \cdot \frac{(q(x)-p(x))^2}{r^2 p^2(x)} \, d\mu(x) \\ &\le \frac{1}{2} \int_\chi [(R-r)p(x) - |q(x)-p(x)|] \frac{(p(x)-q(x))^2}{r^2 p^2(x)} \, d\mu(x) \\ &\le \frac{1}{2} [(R-r) - D_v(p,q)] \cdot \frac{(R-r)^2}{r^2},\end{aligned} \tag{5.8}$$

where $D_v(p,q)$ is the variational distance, i.e., we recall that

$$D_v(p,q) := \int_\chi |p(x) - q(x)| \, d\mu(x),$$

and $L(\cdot, \cdot)$ is the logarithmic mean.

PROOF. We multiply (5.7) by $(d-c)^2 \ge 0$ to obtain

$$\begin{aligned}&\left| (\ln d - \ln c)(d-c) - \frac{(d-c)^2}{L(a,b)} \right| \\ &\le \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{(b-a) - |d-c|} \right)^2 \right] [(b-a) - |d-c|] \cdot \frac{(d-c)^2}{a^2} \\ &\le \frac{1}{2} [(b-a) - |d-c|] \cdot \frac{(d-c)^2}{a^2} \\ &\le \frac{1}{2} [(b-a) - |d-c|] \cdot \frac{(b-a)^2}{a^2}.\end{aligned} \tag{5.9}$$

We choose in (5.9) $a = r$, $b = R$, $c = 1$, and $d = q(x)/p(x)$ to get

$$\begin{aligned}&\left| (\ln q(x) - \ln p(x))(q(x) - p(x)) - \frac{(q(x)-p(x))^2}{p(x) L(r,R)} \right| \\ &\le \left[\frac{1}{4} + \left(\frac{((r+R)/2)p(x) - (p(x)+q(x))/2}{(R-r)p(x) - |q(x)-p(x)|} \right)^2 \right] \\ &\quad \times [(R-r)p(x) - |q(x)-p(x)|] \cdot \frac{(q(x)-p(x))^2}{r^2 p^2(x)} \\ &\le \frac{1}{2} [(R-r)p(x) - |q(x)-p(x)|] \cdot \frac{(p(x)-q(x))^2}{r^2 p^2(x)} \\ &\le \frac{1}{2} [(R-r)p(x) - |q(x)-p(x)|] \cdot \frac{(R-r)^2}{r^2}.\end{aligned}$$

Integrating over x on χ and taking into account the facts that

$$\int_\chi \frac{(q(x)-p(x))^2}{p(x)} \, d\mu(x) = D_{\chi^2}(p,q) \qquad \text{and} \qquad \int_\chi |p(x) - q(x)| \, d\mu(x) = D_v(p,q),$$

we deduce (5.8). ∎

6. APPLICATION TO THE SAMPLING OF CONTINUOUS STREAMS

In monitoring the quality of continuous streams which are common in major sections of the chemical industry, samples of product are collected regularly and analysed. On the basis of these results, the process is allowed to continue operating under existing parameter values or is adjusted in some way.

These results also, when accumulated over a particular production run, can be used to assess the mean quality of the product for the duration of production. If $x(t)$ represents the quality of the stream at time t, then the mean quality for the production time $[0, T]$ is given by $(1/T)\int_0^T x(t)\,dt$.

If the product is a liquid or a gas, it can invariably be sampled instantaneously and so, over the duration of the production period, 'n' test values (say) are available to estimate the mean quality of the stream. These are $x_1, x_2, \dots, x_n$. It is logical then to use the mean of these values, $\bar{x}_n$, to estimate $(1/T)\int_0^T x(t)\,dt$, giving the estimation error as $|(1/T)\int_0^T x(t)\,dt - \bar{x}_n|$.

In some continuous stream processes, however, the product, rather than being purely liquid or gaseous, consists of fine grains of product suspended in a fast moving hot gas stream. Whilst the product is eventually accumulated separate of the carrier gas stream and further processed for ease of handling, it is frequently desired to sample the product whilst it is being manufactured in its suspended state. Under such circumstances, the sample collection time cannot be considered to be instantaneous and, if sampling is being conducted at regular intervals, the collection time may well occupy a significant proportion of the time between the commencement of the collection of consecutive samples.

Suppose that the collection of a sample takes a time p and that the sample thus obtained represents the mean quality of the stream over the time taken for collection. From the perspective of using this single sample to estimate the mean flow over a longer time period which includes this interval, we are led to consideration of the estimation error

$$\left| \frac{1}{d}\int_0^d x(t)\,dt - \frac{1}{p}\int_h^{h+p} x(u)\,du \right|,$$

where $(h, h+p) \subset (0, d)$.

Using inequality (2.3), we may state that

$$\left| \frac{1}{d}\int_0^d x(t)\,dt - \frac{1}{p}\int_h^{h+p} x(u)\,du \right| \le \left[\frac{1}{4} + \left(\frac{1}{2} - \frac{h}{d-p}\right)^2\right](d-p)\,\|x'\|_\infty \le \frac{1}{2}(d-p)\,\|x'\|_\infty, \quad (6.1)$$

provided that $x(\cdot)$ is absolutely continuous on $[0, d]$.

From (6.1), we observe that the best estimate we can get from (6.1) is for $h = (d-p)/2$, obtaining

$$\left| \frac{1}{d}\int_0^d x(t)\,dt - \frac{1}{p}\int_{(d-p)/2}^{(d+p)/2} x(u)\,du \right| \le \frac{1}{4}(d-p)\,\|x'\|_\infty. \quad (6.2)$$

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